

a timelike world line can be computed from the line element (4.8) and (4.12) as

$$\tau_{AB} = \int_A^B d\tau = \int_A^B [dt^2 - (dx^2 + dy^2 + dz^2)/c^2]^{1/2}, \quad (4.13a)$$

$$= \int_{t_A}^{t_B} dt \left\{ 1 - \frac{1}{c^2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] \right\}^{1/2}. \quad (4.13b)$$

More compactly,

$$\tau_{AB} = \int_{t_A}^{t_B} dt' \left[1 - \vec{V}^2(t')/c^2 \right]^{1/2}. \quad (4.14)$$

Time Dilation

The proper time τ_{AB} is *shorter* than the interval $t_B - t_A$ because $\sqrt{1 - \vec{V}^2/c^2}$ is *less* than unity. That is the phenomenon of *time dilation* summarized imperfectly by the slogan “moving clocks run slow.” For time intervals Δt short enough that the velocity $\vec{V}$ is approximately constant over them, it will frequently be useful to make use of the differential form of (4.14),

$$d\tau = dt \sqrt{1 - \vec{V}^2/c^2}. \quad (4.15)$$

Figure 4.11 illustrates the connection.

It should be emphasized that (4.14) or (4.15) hold even for accelerating clocks, i.e., when the velocity is dependent on time.⁹ A famous test of this relation for an accelerating clock is described in Box 4.4 on p. 64.

Example 4.2. A Model Clock. The preceding discussion of time dilation did not refer to the workings of any particular clock. Time dilation is consequence of the geometry of spacetime, and all one needs to know about a clock is that it is a device for measuring the distance along timelike curves. Nevertheless, it is instructive to see how time dilation emerges from the workings of a clock, and the model illustrated in Figures 4.6 and 4.7 provides a simple example. The clock mechanism is the bouncing light pulse. The successive returns of the light pulse to the lower mirror are the events defining the successive intervals along the clock's world line (cf. Figure 4.11). The proper time interval $\Delta\tau$ between these events is the time interval between them in the rest frame of the clock, which is $\Delta\tau \equiv \Delta t = 2L/c$ [cf. (4.3)]. The time interval $\Delta t'$ in the frame where that clock

⁹Occasionally one encounters the misconception that special relativity can deal only with motion at constant velocity. Nothing could be further from the truth. This mistaken idea possibly stems from the fact that inertial frames can differ by uniform motion but not accelerated motion. But this is equally true in Newtonian mechanics, which is mainly concerned with explaining accelerated motion. The high-speed motion of particles in high-energy accelerators is an everyday example of accelerated motion described by the principles of special relativity, as Box 4.4 on p. 64 illustrates.

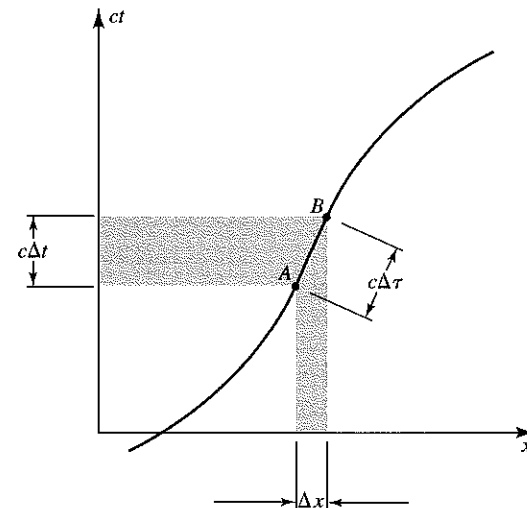


FIGURE 4.11 Proper time and coordinate time. The curve in this figure is the world line of a particle moving in the x -direction. A clock carried with the particle measures the proper time along the world line, which is the spacetime distance along the world line in time units. The proper time, $\Delta\tau$, between two points in spacetime A and B separated by small coordinate intervals Δt and Δx is given by the line element of flat spacetime, (4.8) and (4.12). The interval Δt is longer than $\Delta\tau$; that is time dilation. Judged by the Euclidean geometry of the plane, Δt appears shorter than $\Delta\tau$. But the geometry of a (ct, x) slice of flat spacetime is not Euclidean.

is moving with speed V can be found by eliminating $\Delta x'$ between the first two relations in (4.4). The result is

$$\Delta\tau = \Delta t' (1 - V^2/c^2)^{1/2}. \quad (4.16)$$

This is just the differential relation (4.15) for a clock with speed V . This model clock exhibits time dilation explicitly.

The Twin Paradox

Equation (4.14) shows that the time registered by a clock moving between two points in space depends on the route traveled even if it returns to the same point it started from. This is the source of the famous twin paradox.

Two twins, Alice and Bob, start from rest at one point in space at time t_1 in an inertial frame, as illustrated in Figure 4.12. Alice moves away from the starting point but later returns to rest at the same point at time t_2 . Bob remains at rest at the starting point. The time elapsed on Bob's clock is $t_2 - t_1$. The time elapsed on Alice's clock is always *less* than this because $(1 - \vec{V}^2/c^2)^{1/2}$ is always less than 1 in (4.14). The moving twin ages less than the stationary twin.

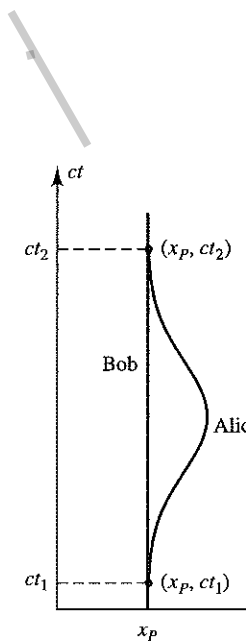


FIGURE 4.12 The twin paradox from a spacetime point of view. Alice and Bob follow two different world lines between the same spacetime points. The lengths of these curves are different, and consequently the proper time registered by clocks carried along each is different.

BOX 4.4 The CERN Muon Lifetime Experiment

Elementary particles that decay into other particles can serve as a kind of clock. The probability of decay is typically an exponential decay law. The time of decay of any particular particle is uncertain. But in a collection of many particles, a fraction $\exp(-t/\tau_p)$ will have decayed after a time t , where the lifetime τ_p is a property of the kind of elementary particle.

Elementary particles can reach velocities close to the velocity of light in particle accelerators. Special relativity predicts that the lifetime of rapidly moving particles should be longer than their lifetime at rest by a factor of $\gamma \equiv (1 - \vec{v}^2/c^2)^{-1/2}$ [cf. (4.14)]. If $\tau_p(\gamma)$ is the lifetime of a particle moving with a speed corresponding to the value of γ , then

$$\tau_p(\gamma) = \gamma \tau_p(1),$$

where $\tau_p(1)$ is the lifetime at rest. Observations of particle decays in accelerators can thus test time dilation.

A particularly accurate test was carried out at CERN in the late 1970s using a special muon storage ring (Bailey et al. 1977). Muons ($\mu^\pm$) are elementary particles having either positive or negative charge. They decay into

neutrinos and electrons or positrons (depending on their charge) with a lifetime of about $2.2 \mu\text{s}$.

Muons circulated in the storage ring in 14-m circular orbits with a measured γ of 29.3 corresponding to $V/c = .9994$. The lifetime of circulating muons of both charges was measured by detecting the electron or positron decay products in counters surrounding the ring. The number of electrons or positrons was monitored as a function of time and fit to a decay law parametrized by the lifetimes $\tau_\mu^\pm$ and a number of other parameters affecting the decay, most importantly the muon magnetic moment. The results were $\tau_\mu^+ = 64.419 \pm .058 \mu\text{s}$ and $\tau_\mu^- = 64.368 \pm .029 \mu\text{s}$. The lifetimes at rest that would be inferred from time dilation, $\tau_\mu^\pm(\gamma)/\gamma$, were compared with independent measurements of the muon lifetime at rest, $\tau_\mu^\pm(1)$. The best results were for the μ^+ 's:

$$[\tau_\mu^+(1) - \tau_\mu^+(\gamma)/\gamma]/\tau_\mu^+(1) = (2 \pm 9) \times 10^{-4}.$$

This is in excellent agreement with the predictions of special relativity. Even an estimate on the basis of the Newtonian formula V^2/R shows that the centripetal acceleration of the muons is large ($\sim 10^{18} \text{ cm/s}^2$), giving good evidence that there is no dependence of time dilation on acceleration.

Example 4.3. Alice accelerates instantaneously to a uniform speed $\frac{4}{5}c$, travels in a straight line away from Bob, eventually instantaneously reverses direction, returns to Bob with the same speed, and decelerates instantaneously to rest. Bob has aged by 50 yr. By how much has Alice aged?

The ages of each are the proper times along their respective world lines between departure and return calculated with (4.14). To understand the contribution of an instantaneous acceleration, first suppose acceleration is uniform over a small time interval of length 2ϵ and take the limit as ϵ vanishes. Then $V = \frac{4}{5}c(t_{\text{mid}} - t)/\epsilon$ between a time ϵ before the midpoint time, t_{mid} , and a time ϵ afterwards. The contribution of this interval to the integral in (4.14) will be proportional to ϵ and negligible as ϵ approaches zero. The same is true for the other two of Alice's accelerations. Thus, the moving clock is running at a rate $[1 - (4/5)^2]^{1/2} = 3/5$ times slower than the stationary clock for 50 yr. Alice will have aged by 30 yr.

The American Heritage dictionary defines *paradox* as “a seemingly contradictory statement that may nonetheless be true.” We obtain a paradox by describing the situation from Alice's point of view. Bob moves away at uniform speed, re-

verses direction, and returns at uniform speed. That seems to be exactly the same as the situation from Bob's point of view, who sees Alice move away at uniform speed, reverse direction, and return at uniform speed. The result is not symmetric; Alice is younger than Bob.¹⁰ However, their situations are not symmetric. Alice and Bob travel two *different* world lines in spacetime with *different* distances between their starting and ending points. Their clocks measure these distances and so read differently.

Straight Lines and Longest Distances

The twin paradox example illustrates an important property of the non-Euclidean geometry of flat spacetime. As (4.14) shows, *every* distinct timelike world line that Alice could follow between points A and B has a shorter length than the straight line curve followed by Bob. (Other curves may look longer in a figure like Figure 4.12, but are in fact shorter because the geometry is non-Euclidean. Recall Example 4.1.) The straight line path is the *longest* distance between any two timelike separated points in flat four-dimensional spacetime.¹¹ To see this, pick any two timelike separated points A and B . The straight-line path between them is a world line moving with some constant velocity $\vec{V}$. Use that velocity to transform to another inertial frame where the two events occur at the same *place*. That frame is like Bob's discussed above. Any observer like Alice moving on a non-straight path measures a *shorter* spacetime distance between the events than Bob does. Spacetime distances don't depend on the inertial frame used to calculate them in. In three-dimensional space a straight line is the shortest distance between any two points, but in flat spacetime a straight line is the longest distance between two timelike separated points.

4.5 Lorentz Boosts

The Connection Between Inertial Frames

The discussion of the construction of inertial frames in both Newtonian mechanics, Section 3.1, and special relativity, Section 4.3, shows that two inertial frames can differ from one another by rotations, displacements, and uniform motions (or combinations thereof). Rotations and displacements work in the same way as in Newtonian mechanics, but let's now find the transformation associated with uniform motion that generalizes the Galilean transformation (3.6) to special relativity.

The line element (4.8) specifies the geometry of special relativistic spacetime in terms of four rectangular coordinates (t, x, y, z) defining an inertial frame—

¹⁰For a direct experimental test of the twin paradox in the slightly curved spacetime of the Earth, see Box 6.2 on p. 130.

¹¹A straight-line path in curved spacetime is not always a path of longest proper time, but it is a path of extremal proper time. That is discussed on p. 131.