

1.13 Appendix: A full discussion of the twin 'paradox'

The problem

Diana leaves her twin Artemis behind on Earth and travels in her rocket for 2.2×10^8 s (≈ 7 yr) of *her* time at $24/25 = 0.96$ the speed of light. She then instantaneously reverses her direction (fearlessly braving those gs) and returns to Earth in the same manner. Who is older at the reunion of the twins? A spacetime diagram can be very helpful.

Brief solution. Refer to Fig. 1.16 below. Diana travels out on line $\mathcal{PB}$. In her frame, Artemis' event $\mathcal{A}$ is simultaneous with event $\mathcal{B}$, so Artemis is indeed ageing slowly. But as soon as Diana turns around she changes inertial reference frames: now she regards $\mathcal{B}$ as simultaneous with Artemis' event $\mathcal{C}$! Effectively, Diana sees Artemis age incredibly quickly for a moment. This one spurt more than makes up for the slowness Diana observed all along. Numerically, Artemis ages 50 years for Diana's 14.

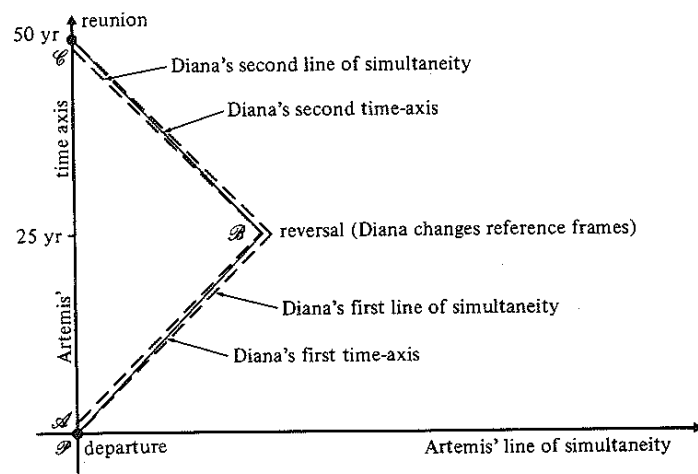


Fig. 1.16 The idealized twin 'paradox' in the spacetime diagram of the stay-at-home twin.

Fuller discussion. For readers who are unsatisfied with the statement 'Diana sees Artemis age incredibly quickly for a moment', or who wonder what physics lies underneath such a statement, we will discuss this in more detail, bearing in mind that the statement 'Diana sees' really means 'Diana observes', using the rods, clocks, and data bank that every good relativistic observer has.

Diana might make her measurements in the following way. Blasting off from Earth, she leaps on to an inertial frame called $\bar{\mathcal{O}}$ rushing away from the Earth at $v = 0.96$. As soon as she gets settled in this new frame she orders all clocks

synchronized with hers, which read $\bar{t} = 0$ upon leaving Earth. She further places a graduate student on every one of her clocks and orders each of them who rides a clock that passes Earth to note the time on Earth's clock at the event of passage. After traveling seven years by her own watch, she leaps off inertial frame $\bar{\mathcal{O}}$ and grabs hold of another one $\bar{\mathcal{O}}$ that is flying *toward* Earth at $v = 0.96$ (measured in Earth's frame, of course). When she settles into this frame she again distributes her graduate students on the clocks and orders all clocks to be synchronized with hers, which read $\bar{t} = 7$ yr at the changeover. (All clocks were already synchronized with each other – she just adjusts only their zero of time.) She further orders that every graduate student who passes Earth from $\bar{t} = 7$ yr until she gets there herself should record the time of passage and the reading of Earth's clocks at that event.

Diana finally arrives home after ageing 14 years. Knowing a little about time dilation, she expects Artemis to have aged much less, but to her surprise Artemis is a wizened old prune, a full 50 years older! Diana keeps her surprise to herself and runs over to the computer room to check out the data. She reads the dispatches from the graduate students riding the clocks of the outgoing frame. Sure enough, Artemis seems to have aged very slowly by their reports. At Diana's time $\bar{t} = 7$ yr, the graduate student passing Earth recorded that Earth's clocks read only slightly less than two years of elapsed time. But then Diana checks the information from her graduate students riding the clocks of the ingoing frame. She finds that at her time $\bar{t} = 7$ yr, the graduate student reported a reading of Earth's clocks at more than 48 years of elapsed time! How could one student see Earth to be at $t = 2$ yr, and another student, *at the same time*, see it at $t = 48$ yr? Diana leaves the computer room muttering about the declining standards of undergraduate education today.

We know the mistake Diana made, however. Her two messengers did *not* pass Earth at the same time. Their clocks read the same amount, but they encountered Earth at the very different events $\mathcal{A}$ and $\mathcal{C}$. Diana should have asked the first frame students to continue recording information until they saw the second frame's $\bar{t} = 7$ yr student pass Earth. What does it matter, after all, that they would have sent her dispatches dated $\bar{t} = 171$ yr? Time is only a coordinate. One must be sure to catch *all* the events.

What Diana really did was use a bad coordinate system. By demanding information only before $\bar{t} = 7$ yr in the outgoing frame and only after $\bar{t} = 7$ yr in the ingoing frame, she left the whole interior of the triangle $\mathcal{ABC}$ out of her coordinate patches (Fig. 1.17(a)). Small wonder that a lot happened that she did not discover! Had she allowed the first frame's students to gather data until $\bar{t} = 171$ yr, she could have covered the interior of that triangle.

One can devise an analogy with rotations in the plane (Fig. 1.17(b)). Consider trying to measure the length of the curve $ABCD$, but being forced to rotate coordinates in the middle of the measurement, say after you have measured from A to B in the x - y system. If you then rotate to $\bar{x}$ - $\bar{y}$, you must resume the measuring

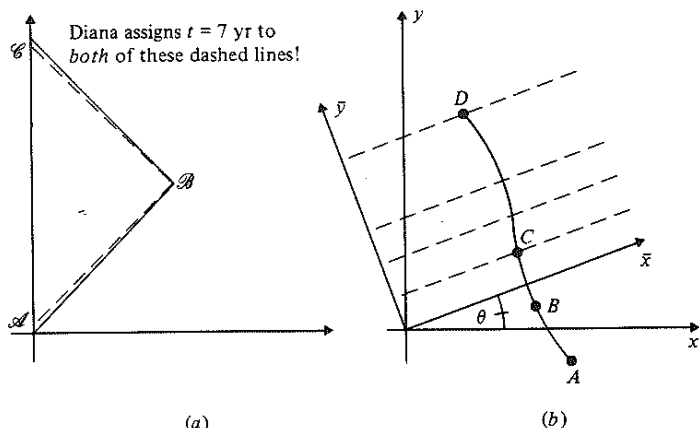


Fig. 1.17 Diana's change of frame is analogous to a rotation of coordinates in Euclidean geometry.

at B again, which might be at a coordinate $\bar{y} = -5$, whereas originally B had coordinate $y = 2$. If you were to measure the curve's length starting at whatever point had $\bar{y} = 2$ (same $\bar{y}$ as the y value you ended at in the other frame), you would begin at C and get much too short a length for the curve.

Now, nobody would make that error in measurements in a plane. But lots of people would if they were confronted by the twin paradox. This comes from our refusal to see time as simply a coordinate. We are used to thinking of a universal time, the same everywhere to everyone regardless of their motion. But it is not the same to everyone, and one must treat it as a coordinate, and make sure that one's coordinates cover all of spacetime.

Coordinates that do not cover all of spacetime have caused a lot of problems in GR. When we study gravitational collapse and black holes we will see that the usual coordinates for the spacetime outside the black hole do not reach inside the black hole. For this reason, a particle falling into a black hole takes infinite coordinate time to go a finite distance. This is purely the fault of the coordinates: the particle falls in a finite proper time, into a region not covered by the 'outside' coordinates. A coordinate system that covers both inside and outside satisfactorily was not discovered until the mid-1950s.

1.14 Exercises

- 1 Convert the following to units in which $c = 1$, expressing everything in terms of m and kg:
 - (a) Worked example: 10 J. In SI units, $10 \text{ J} = 10 \text{ kg m}^2 \text{ s}^{-2}$. Since $c = 1$, we have $1 \text{ s} = 3 \times 10^8 \text{ m}$, and so $1 \text{ s}^{-2} = (9 \times 10^{16})^{-1} \text{ m}^{-2}$. Therefore we get